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1981 J. Phys. A: Math. Gen. 14 981

(http://iopscience.iop.org/0305-4470/14/4/025)

J. Phys. A: Math. Gen. 14 (1981) 981–992. Printed in Great Britain

Domain wall singularity in a Landau–Ginzburg model

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Received 12 August 1980

Abstract. We study the interface between two coexisting phases in an anisotropic, $(n+1)$-component Landau–Ginzburg model, with a single easy axis, and $O(n)$ symmetry in the transverse components, which may serve to represent a d-dimensional uniaxial ferromagnet below its Curie point, T_c. An exact solution of the mean-field equations of motion due to Sarker *et al* indicates the existence of a bifurcation temperature, $T_B < T_c$. For $T < T_B$, the interface profile is consistent with the usual Bloch picture, with a non-zero density of transverse magnetisation within the wall. For $T > T_B$, the profile is Ising-like, with no transverse magnetisation, and approaches the known universal profile as $T \to T_c$. Analyses of fluctuations about this solution shows that the bifurcation is quite analogous to a second-order phase transition. The amplitude of transverse magnetisation vanishes as $T \to T_B^-$ and an associated susceptibility diverges with the exponents of the $(d-1)$-dimensional n-vector model.

1. Introduction

Theoretical studies of the interface between two coexisting phases just below a critical point have, to date, been mainly concerned with Ising-like systems. In the context of field-theoretic models of the Landau–Ginzburg type, which commonly serve as a starting point for renormalisation-group analysis of critical phenomena in the bulk, these investigations are facilitated by the existence of the well known 'kink' solution, $\phi \propto \tanh \lambda z$ of ϕ^4 theory in the mean-field approximation, where the interfacial width, λ^{-1}, is proportional to the bulk correlation length. Analyses of fluctuation corrections to this solution by various authors (Rudnick and Jasnow 1978a, b, Ohta and Kawasaki 1977, see also Wallace and Zia 1979) yield results for the interfacial profile which are in good agreement with experimental measurements on liquid–gas systems (Huang and Webb 1969, Wu and Webb 1973).

The notion of universality of critical behaviour in the bulk is well supported by both experimental evidence and theoretical considerations. Thus, for example, it appears that the asymptotic critical behaviour both of simple fluids and of uniaxial magnets are derivable from the same Landau–Ginzburg model. However, it is less clear to what extent the properties of interfaces are also universal. The example we have in mind is that of a ferromagnet with a single easy axis of magnetisation. For such a system, the analogue of the gas–liquid profile, which seems to be universal for spatial dimensionality $d > 3$, with small non-universal corrections when $d = 3$ (Rudnick and Jasnow 1978b), would be a state in which the magnetisation density is always parallel to the easy axis, but varies continuously in amplitude, with a change of sign at the middle of the

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interface. The interfacial thickness in such a state would be proportional to the bulk correlation length. This picture would appear, at first sight, to be at variance with the elementary Bloch model of a domain wall, according to which, spins within the wall are gradually bent away from the easy axis, and the wall thickness is determined not by the correlation length, but by the ratio of the anisotropy and exchange energies.

To examine this question in more detail, we report in this paper an investigation of the interface in an anisotropic Landau–Ginzburg model which should, at least for temperatures sufficiently close to the Curie point, serve as a model for the domain wall in a uniaxial ferromagnet. For simplicity, we take the easy axis to lie parallel to the interface. From elementary magnetostatic considerations, the component of magnetisation normal to the interface is then required to remain constant, and therefore equal to zero. For a system of three-component spins, we are left with two independent components; the bulk spatial dimensionality, d, is taken as usual to be an independent parameter. It is in fact instructive to consider the more general model defined by the Hamiltonian

$$\mathcal{H} = \int d^d x\{\tfrac{1}{2}|\nabla S_1|^2 + \tfrac{1}{2}|\nabla \boldsymbol{S}_2|^2 - \tfrac{1}{2}r(S_1^2 + \boldsymbol{S}_2^2) + g\boldsymbol{S}_2^2 + u(S_1^2 + \boldsymbol{S}_2^2)^2/4!\}, \tag{1.1}$$

where $S_1(x)$ is a scalar field, and $\boldsymbol{S}_2(x)$ is an n-component vector. The parameter g here is proportional to the ratio of the anisotropy and exchange energies. Although this quantity is in general significantly temperature dependent, it is not expected to change sign at the critical point, and for the purpose of this model may be taken as an independent variable. The most significant temperature dependence appears as usual in r, which may be taken as linear in temperature, while u may be taken as a positive constant.

The bulk critical properties of this model have been studied extensively (see e.g. Fisher and Pfeuty 1972, Nelson and Domany 1976, Amit and Goldschmidt 1978). Its mean-field phase diagram is shown in figure 1, in terms of g and

$$t = (g - r) \propto T - T_0(g) \tag{1.2}$$

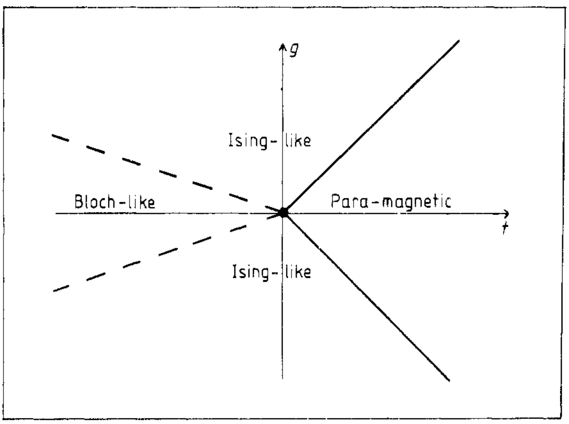

Figure 1. Phase diagram in the mean-field approximation. Loci of critical points (full lines) and bifurcation points (broken lines) meet at the bicritical point. For negative g, the bifurcation locus is present only when $n = 1$, there being no interface with a broken continuous symmetry.

where $T_0(g)$ is the mean-field transition temperature; the two critical loci at $t = |g|$ meet at the bicritical point $t = g = 0$. For the case of a single easy axis of magnetisation, we take $g \geqslant 0$ in what follows. The magnetisation in the ordered phase, $t < |g|$, is found in mean-field theory by solving the equations

$$\delta\mathcal{H}/\delta S_1(x) = -\nabla^2 S_1 - rS_1 + \tfrac{1}{6}uS_1(S_1^2 + S_2^2) = 0 \tag{1.3}$$

$$\delta\mathcal{H}/\delta \boldsymbol{S}_2(x) = -\nabla^2 \boldsymbol{S}_2 + (2g - r)\boldsymbol{S}_2 + \tfrac{1}{6}u\boldsymbol{S}_2(S_1^2 + S_2^2) = 0. \tag{1.4}$$

The uniform solutions are given, for $g > 0$, by

$$S_1 = \pm(6r/u)^{1/2}, \qquad \boldsymbol{S}_2 = 0, \tag{1.5}$$

and represent absolute minima of (1.1). A pair of solitary wave solutions which interpolate between these two minima, and in which the spin fields vary only in one spatial direction (the z axis, say), has been given in another context by Sarker *et al* (1976). For all positive values of r and g, there is a solution

$$S_1 = \pm(6r/u)^{1/2} \tanh \lambda z, \tag{1.6}$$

$$\boldsymbol{S}_2 = 0,$$

with

$$\lambda^2 = \tfrac{1}{2}r. \tag{1.7}$$

However, this solution is stable (i.e. it is a local minimum of (1.1)) only when $r \leqslant 4g$. For $r > 4g$, there is a second, stable solution, namely

$$S_1 = \pm(6r/u)^{1/2} \tanh \lambda z, \tag{1.8}$$

$$\boldsymbol{S}_2 = [6(r - 4g)/u]^{1/2} (\operatorname{sech} \lambda z)\boldsymbol{s} \tag{1.9}$$

where

$$\lambda^2 = 2g, \tag{1.10}$$

and $\boldsymbol{s}$ is an arbitrary n-component unit vector.

Evidently, the first solution, appropriate for temperatures sufficiently close to the critical temperature, or for sufficiently large anisotropy, corresponds to the usual interface in a purely Ising-like system. Indeed, one may use standard methods to evaluate fluctuation corrections and, at least to first order in the ε_b expansion (where ε_b denotes the deviation of the bulk dimensionality from 4) one obtains a result which differs from that of Rudnick and Jasnow (1978a) only by a minor correction to the interfacial width. One sees from (1.7) that this width is again proportional to the bulk correlation length. The second solution, appropriate for sufficiently low temperatures, or small enough anisotropy, is consistent with the usual Bloch wall picture: in the case $n = 1$ one has a local magnetisation density, of magnitude

$$M(z) = (S_1^2 + S_2^2)^{1/2}, \tag{1.11}$$

which makes an angle $\theta(z)$ with the anisotropy axis, given by

$$\tan \theta = |\boldsymbol{S}_2|/S_1. \tag{1.12}$$

At sufficiently low temperatures, $r \gg 4g$, the magnetisation density (1.11) is approximately constant, and equal to the spontaneous magnetisation of the bulk phases, while $\theta(z)$ is given approximately by

$$\cos \theta(z) = \tanh \lambda z. \tag{1.13}$$

In this low-temperature state, the width of the domain wall is determined entirely by the anisotropy parameter, g, according to (1.10).

For reasons which will become apparent in the next section, we have not succeeded in evaluating fluctuation corrections to the profile given by (1.8) and (1.9). Indeed, the result of such a calculation would probably have no universal significance since this profile does not appear in the immediate neighbourhood of the critical loci. The purpose of the work reported here is to elucidate the nature of the singularity which occurs on the loci $r=4|g|$, or $t=-3|g|$, indicated by the broken lines in figure 1. Following Sarker *et al* (1976) we refer to these lines as loci of bifurcation points, which seems to be consistent with the terminology of non-linear differential equation theory. In particular, we suggest that this singularity has the hallmarks of a second-order phase transition occurring within the interfacial region. For $r<4|g|$, which we call the 'Ising-like' region, the sign of the solution (1.5) will be selected either spontaneously or by application of appropriate boundary conditions at $z=\pm\infty$. For definiteness we choose the positive sign, which commits us also to choosing the positive sign in (1.8) for $r<4|g|$. This latter region we refer to as 'Bloch-like'. What happens as one approaches a bifurcation point from the Bloch-like side is that the local minima of the Hamiltonian, or free energy functional, (1.1), corresponding to the unit vector $\boldsymbol{S}$, coalesce in a continuous manner, leading to a single local minimum (1.6) in the Ising-like region. In the usual way, one may identify an order parameter, namely the amplitude of the transverse magnetisation $\boldsymbol{S}_2$, which is equal to zero in the Ising-like region, and which, in the Bloch-like region, varies as $(r-4g)^{\beta_B}$, with the usual mean-field-like exponent

$$\beta_B=\tfrac{1}{2}, \tag{1.14}$$

the subscript denoting bifurcation. Furthermore, as we shall see in the next section, there is a susceptibility, associated with fluctuations in $\boldsymbol{S}_2$, which diverges at the bifurcation loci as $|r-4g|^{-\gamma_B}$, again with the mean-field-like exponent

$$\gamma_B=1 \tag{1.15}$$

in this approximation.

In the following section we pursue this analogy in detail, and use standard semiclassical methods (Gervais and Sakita 1975, Gervais *et al* 1976) to derive the effective Hamiltonian governing the transition.

Since the transition occurs essentially in a spatial region of finite extent in the z direction, we may anticipate the existence of a borderline bulk dimensionality, $d^\times=5$, above which the mean-field treatment of this section is adequate, and this expectation will be confirmed. One might anticipate further that the leading singularity will be that of the $(d-1)$-dimensional n-vector model: indeed a conjecture to this effect has recently been advanced by Lajzerowicz and Niez (1979) for the case $n=1$. We shall show that this picture is essentially correct. However, this result is far from trivial, owing to the presence of a massless Goldstone mode associated with the spontaneous breaking of the Euclidean invariance by the interface. The singularities induced by such modes in isolation have been studied in some detail (Wallace and Zia 1979, Lowe and Wallace 1980). The effective Hamiltonians governing such modes are of a 'surface tension' type.

The important feature here is that the Goldstone mode apparently couples to the order parameter of the bifurcation transition, and might be expected to modify the leading infrared singularity. In fact, however, as we show, a correct treatment of the

Goldstone mode yields only effective couplings with sufficiently high derivatives to ensure their irrelevance in the usual renormalisation-group sense.

The final topic to be considered in this section is the shape of the bifurcation loci near the bicritical point, $t = g = 0$. As is well known, the effect of critical fluctuations on the shape of the critical loci, $t = t_c(g)$, is pronounced, and leads to the cusp at the bicritical point shown schematically in figure 2. Renormalisation-group analysis indicates that the critical loci near the bicritical point have the form

$$t_c(g) = \text{constant}\,|g|^{1/\phi} \tag{1.16}$$

where the crossover exponent is given (Wilson 1972) by

$$\phi = 1 + (n+1)\varepsilon_b/2(n+9) + O(\varepsilon_b^2); \qquad \varepsilon_b = 4 - d. \tag{1.17}$$

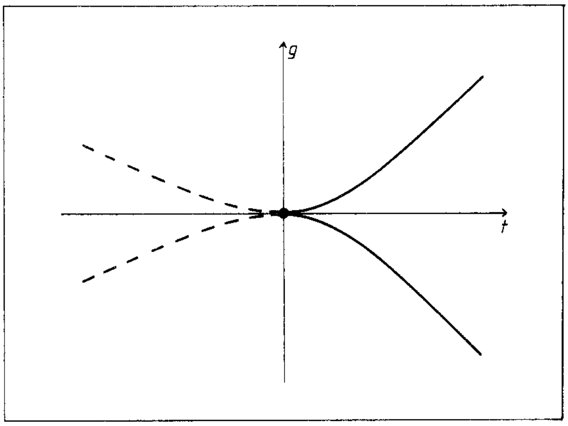

Figure 2. Schematic representation of the phase diagram including the effects of critical fluctuations.

The following argument, of a standard type, indicates that the shape of the bifurcation loci is governed by the *same* crossover exponent (1.17). The free energy of the system may be expanded as a Taylor series in the magnetisation, namely

$$F[\{M_i(x)\}g, t] = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{i_1 \ldots i_m = 1}^{n+1} \int dx_1 \ldots dx_m\, \Gamma_{i_1 \ldots i_m}(x_1 \ldots x_m; g, t) M_{i_1}(x_1) \ldots M_{i_m}(x_m), \tag{1.18}$$

where

$$M_i(x) = \langle S_i(x) \rangle, \tag{1.19}$$

and the vertex functions, Γ, are those appropriate to the bulk phases. (The uninitiated reader is referred to the book by Amit (1978) for standard field-theoretic methods and terminology.) The full interfacial profile is obtained, in principle, by minimising, with respect to $M_i(x)$, the free energy (1.18) which reduces, in the mean-field approximation, to (1.1). General scaling arguments and direct renormalisation-group analysis show that there is a region near the bicritical point, in which the vertex functions are well approximated by the scaling form

$$\Gamma_\alpha(x_i; g, t) \approx |t|^{\lambda_\alpha} \tilde{\Gamma}_\alpha(|t|^{\nu} x_i; g|t|^{-\phi}) \tag{1.20}$$

where α represents the collection of labels appearing in (1.18), and the exponents λ_α, the correlation length exponent ν and the crossover exponent ϕ are those characteristic of the bicritical point, rather than the critical loci. Although (1.20) yields no direct information about the nature of singularities from the bicritical point, one may conclude that such singularities must occur, if they occur at all, when the argument $g|t|^{-\phi}$ reaches some critical value, and this leads directly to (1.16) for the location of the critical loci.

Now, (1.20), together with explicit results for λ_α, leads to a similar scaling form for the free energy, namely

$$F[\{M_i(x)\}, g, t] \approx |t|^{-d\nu} \tilde{F}[\{\tilde{M}_i(|t|^\nu x)\}, g|t|^{-\phi}] \tag{1.21}$$

where $\tilde{M}_i$ is defined by

$$M_i(x) = |t|^\beta \tilde{M}_i(|t|^\nu x) \tag{1.22}$$

and β is again the order parameter exponent appropriate to the bicritical point. Minimisation of (1.20) implies that $\tilde{M}_i$ has the form

$$\tilde{M}_i = \tilde{M}_i(|t|^\nu x, g|t|^{-\phi}), \tag{1.23}$$

and, provided that M_i depends smoothly on x, we conclude that any singularity in M_i, whether of the critical or bifurcation variety, occurs when $g|t|^{-\phi}$ reaches an appropriate value. Thus, the form of the bifurcation loci is that indicated in figure 2. It is perhaps worth emphasising that (1.22) does not in itself yield any information about the nature of the bifurcation singularity. In particular, it does not imply that the interfacial width has the form constant $|t|^{-\nu}$, as may be verified by expressing (1.8) and (1.9) in the form (1.22) with $\nu = \frac{1}{2}$ and $\phi = 1$.

From now on, we shall be concerned with the nature of the bifurcation singularity away from the bicritical point.

2. Effective Hamiltonian for the bifurcation singularity

In this section we analyse the effect of thermal fluctuations of the mean-field-like description of the bifurcation transition given in § 1. In the usual way, we shift the fields S_i by their classical expectation values, writing

$$S_i(\underset{\sim}{x}, z) = M_i(z+q) + \phi_i(\underset{\sim}{x}, z+q) \tag{2.1}$$

where $\underset{\sim}{x}$ denotes the $(d-1)$ coordinates parallel to the interface, M_i denotes the classical solution (1.6) or (1.8) and (1.9) and q is at this stage a constant parameter locating the position of the interface. Fluctuations in the deviation fields ϕ_i are then governed by the partition function

$$Z = \int \mathrm{d}[\phi] \exp\{-\mathscr{H}(\phi)\} \tag{2.2}$$

where $\int \mathrm{d}[\phi]$ denotes functional integration and $\mathscr{H}(\phi)$ is obtained by substituting (2.1) in (1.1). In the Ising-like region, $r < 4g$, this yields

$$\mathscr{H}(\phi) = \text{constant} + \int \mathrm{d}^d x \{\tfrac{1}{2}\phi_1 D_1 \phi_1 + \tfrac{1}{2}\boldsymbol{\phi}_2 \cdot D_2 \boldsymbol{\phi}_2 + \mathrm{O}(\phi_i^3)\} \tag{2.3}$$

where the differential operators D_1 and D_2 are given by

$$D_1 = -\nabla^2 + 2\lambda^2(2 - 3\,\mathrm{sech}^2\,\lambda z) \tag{2.4}$$

$$D_2 = -\nabla^2 + \lambda^2(1 - 2\,\mathrm{sech}^2\,\lambda z) + \sigma \tag{2.5}$$

with

$$\sigma = \tfrac{1}{2}(4g - r). \tag{2.6}$$

To obtain propagators for the ϕ fields, one must diagonalise the quadratic part of (2.3) by solving the eigenvalue equation

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} u_\alpha & (x) \\ \boldsymbol{v}_\alpha & (x) \end{pmatrix} = \mu_\alpha \begin{pmatrix} u_\alpha & (x) \\ \boldsymbol{v}_\alpha & (x) \end{pmatrix}. \tag{2.7}$$

The solutions we require explicitly are soft modes, namely those which have zero eigenvalues when $\sigma = 0$. There are two of these. One is the expected translation mode given, up to an appropriate normalisation, by

$$\begin{pmatrix} u_0 \\ \boldsymbol{v}_0 \end{pmatrix} = \begin{pmatrix} \mathrm{sech}^2\,\lambda(z+q) \\ 0 \end{pmatrix} \exp(\mathrm{i}\underset{\sim}{k}\cdot\underset{\sim}{x}); \qquad \mu_0 = k^2, \tag{2.8}$$

which reduces to the gradient of (1.6) when $k^2 = 0$. The other is

$$\begin{pmatrix} u_\sigma \\ \boldsymbol{v}_\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{v}\,\mathrm{sech}\,\lambda(z+q) \end{pmatrix} \exp(\mathrm{i}\underset{\sim}{k}\cdot\underset{\sim}{x}); \qquad \mu_\sigma = \sigma + k^2, \tag{2.9}$$

where $\boldsymbol{v}$ is a constant vector.

Derivation of the corresponding modes in the Bloch-like state is discussed in the Appendix.

One may now envisage expanding ϕ_1 and $\boldsymbol{\phi}_2$ in the eigenfunctions u_α and $\boldsymbol{v}_\alpha$, the expansion coefficients becoming the new functional integration variables. Evidently, the coefficient $\boldsymbol{\chi}(k)$ of (2.9) in this expansion represents the order parameter of the transition. When σ becomes negative, $\boldsymbol{\chi}$ will acquire an expectation value, proportional to $|\sigma|^{1/2}$ at the mean-field level, leading to the new interface profile (1.9). The correlation function associated with fluctuations in $\boldsymbol{\chi}$ is given at the mean-field level by $(\sigma + k^2)^{-1}$, giving a susceptibility exponent $\gamma_B = 1$, when $q^2 = 0$. We denote the remaining terms of the normal mode expansion collectively by

$$\begin{pmatrix} R_1(\underset{\sim}{x}, z+q) \\ \boldsymbol{R}_2(\underset{\sim}{x}, z+q) \end{pmatrix}, \tag{2.10}$$

and evidently have the orthogonality conditions

$$\int u_0^* R_1\,\mathrm{d}x = \int \boldsymbol{v}_\sigma^* \cdot \boldsymbol{R}_2 \mathrm{d}z = 0. \tag{2.11}$$

To obtain the desired effective Hamiltonian it is convenient to replace functional integration over the coefficient of (2.8) by integration over the collective coordinate q, which is now considered a function of the $(d-1)$ spatial coordinates $\underset{\sim}{x}$. The Jacobian associated with the transformations of this kind is well known (see e.g. Diehl *et al* 1980) and may be ignored if one uses a dimensional regularisation scheme to control ultraviolet divergences. We thus substitute into the Hamiltonian (1.1) the expression

$$\begin{pmatrix} S_1 \\ \boldsymbol{S}_2 \end{pmatrix} = \begin{pmatrix} M_1(z+q(\underset{\sim}{x})) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \boldsymbol{\chi}(\underset{\sim}{x})\,\mathrm{sech}\,\lambda(z+q(\underset{\sim}{x})) \end{pmatrix} + \begin{pmatrix} R_1(\underset{\sim}{x}, z+q(x)) \\ \boldsymbol{R}_2(\underset{\sim}{x}, z+q(x)) \end{pmatrix}. \tag{2.12}$$

The result is conveniently written as a sum of three terms:

$$\mathcal{H} = \mathcal{H}_1(q, \chi) + \mathcal{H}_2(R_i) + \mathcal{H}_3(\partial q, \chi, R_i). \tag{2.13}$$

The first term is given by

$$\mathcal{H}_1 = \int \mathrm{d}^{d-1}x\{A(\partial q)^2 + B[\tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\sigma\chi^2] + C\chi^4\}, \tag{2.14}$$

where A, B, C are constants, and ∂ represents the $(d-1)$-dimensional gradient operator. The second has the form

$$\mathcal{H}_2 = \int \mathrm{d}^d x\{\tfrac{1}{2}R_1 D_1 R_1 + \tfrac{1}{2}\boldsymbol{R}_2 \cdot D_2 \boldsymbol{R}_2 + \mathrm{O}(R_i^3)\} \tag{2.15}$$

and, taken in conjunction with the orthogonality conditions (2.11), ensures that R_1 and $\boldsymbol{R}_2$ are massive fields. Finally, the interaction terms in $\mathcal{H}_3$ are linear or quadratic in R_i, and at least quadratic in χ and ∂q.

The required effective Hamiltonian, involving only $\chi(\underset{\sim}{x})$ and the translation mode as represented by $q(\underset{\sim}{x})$, is now obtained by integrating out the R_i and is given by

$$\mathcal{H}_{\mathrm{eff}} = \mathcal{H}_1 + \ln\langle\exp\{-\mathcal{H}_3\}\rangle, \tag{2.16}$$

where angular brackets represent averaging in the ensemble of $\mathcal{H}_2$. It has the form

$$\mathcal{H}_{\mathrm{eff}} = \int \mathrm{d}^{d-1}x\{\tfrac{1}{2}A(\partial q)^2 + B[\tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\sigma\chi^2] + \tilde{C}\chi^4\} + \mathrm{O}((\partial q)^2\chi^2, \chi^6, (\partial q)^4, \text{etc}). \tag{2.17}$$

The terms multiplied by B and $\tilde{C}$ yield the standard $(d-1)$-dimensional n-vector model which, taken alone, has an infrared singularity governed by the isotropic fixed point of the renormalisation group. Furthermore, one sees that interactions with the translation mode are only via operators which are irrelevant at that fixed point, and thus will contribute only higher order corrections. We thus confirm that the exponents of the bifurcation transition should be those of the isotropic, $(d-1)$-dimensional n-vector model.

Since our effective Hamiltonian (2.17) has been derived by considering the high-temperature, Ising-like state, it remains to inquire whether the residual interactions with the translation mode can seriously affect the nature of the low-temperature, Bloch-like state. In the two-component case ($n=1$) these interactions are clearly irrelevant; however, for $n \geqslant 2$, there are additional Goldstone modes due to the spontaneously broken $\mathrm{O}(n)$ invariance, and it is not immediately obvious that the effects of the translation mode will be less relevant than the self-interactions of these modes. This can be confirmed by introducing the $\mathrm{O}(n)$ Goldstone modes as collective coordinates in our expansion about the Bloch wall type interface, by analogy with the introduction of a translation mode collective coordinate in (2.12). An analysis similar to the above achieves this. Explicitly we obtain

$$\mathcal{H}_{\mathrm{eff}} = \int \mathrm{d}^{d-1}x\left\{\tfrac{1}{2}(\partial q)^2 + \sum_{i=1}^{n} \partial\alpha_i\, \partial\alpha_i + \mathrm{O}\left[(\partial q)^2 \sum_{i=1}^{n} \partial\alpha_i\, \partial\alpha_i + \left(\sum_{i=1}^{n} \partial\alpha_i\, \partial\alpha_i\right)^2\right]\right\} \tag{2.18}$$

where

$$\alpha_n^2 = 1 - \sum_{i=1}^{n-1} \alpha_i^2. \tag{2.19}$$

We thus confirm that interactions with the translation mode do not affect the nature of the ordered state in leading order.

3. Discussion

We have shown that the interface in our model system, of bulk dimensionality d, undergoes a second-order phase transition, whose leading singularity is that of the $(d-1)$-dimensional n-vector model, n being the number of spin components transverse to the single ordering axis. Our conclusions for the physically interesting case $d=3$ are best summarised for several distinct cases.

3.1. Two-component model ($n=1$).

In this case, the transition temperature T_B separates a state with a Bloch-like profile for $T<T_B$ whose width is determined by the degree of anisotropy, from a state with an Ising-like profile for $T>T_B$ of width equal to the bulk correlation length. As $T\to T_B^-$, the transverse magnetisation density within the interface vanishes as

$$M_2 \sim (T_B - T)^{\beta_B} F(z) \tag{3.1}$$

where $F(z)\sim \operatorname{sech} \lambda z$ in the mean-field approximation. In addition there is a divergent susceptibility, which may be understood formally by considering the correlation function

$$G_{ij}(\underset{\sim}{x}z;\underset{\sim}{x}'z') = \langle [S_i(\underset{\sim}{x}z) - M_i(z)][S_j(\underset{\sim}{x}'z') - M_j(z')]\rangle. \tag{3.2}$$

Regarded as an integral operator, G_{ij} has a set of eigenfunctions, $u_i(\underset{\sim}{k}\alpha;\underset{\sim}{x}z)$, with eigenvalues $g(\underset{\sim}{k}\alpha;T)$, satisfying

$$\sum_j \int \mathrm{d}^2x'\,\mathrm{d}z'\,G_{ij}(\underset{\sim}{x}z;\underset{\sim}{x}'z')u_j(\underset{\sim}{k}\alpha;\underset{\sim}{x}'z') = g(\underset{\sim}{k}\alpha;T)u_i(\underset{\sim}{k}\alpha;xz), \tag{3.3}$$

where $\underset{\sim}{k}$ is the wavevector in the plane of the interface, and α labels normal modes in the z direction. One of these eigenvalues, $g(\underset{\sim}{k}\sigma;T)$, which is given in mean-field theory by the inverse of μ_σ in (2.9), diverges when $\underset{\sim}{k}=0$ and $T\to T_B$ as

$$g(0\sigma;T)\sim |T-T_B|^{-\gamma_B}. \tag{3.4}$$

Since G_{ij} may be expanded in its eigenfunctions, with expansion coefficients $g(\underset{\sim}{k}\alpha;T)$, this represents a singularity of the correlation function, which should in principle be detectable, for example by neutron scattering. The exponents β_B and γ_B are those of the single-component S^4 model which, according to the usual universality arguments, should be identical with those of the two-dimensional Ising model, namely

$$\beta_B = \tfrac{1}{8}; \qquad \gamma_B = \tfrac{7}{4}. \tag{3.5}$$

3.2. Three-component model ($n=2$).

In the isotropic model that we have considered explicitly, the case $n=2$ yields the singularity of the two-dimensional XY model. Although this singularity cannot be studied directly in the S^4 model, by the expansion around $(d-1)=4$, general universality arguments, supported by transformations of the kind leading to (2.18), indicate that it should have the same asymptotic form as that found, for example, by José *et al* (1977), for systems of fixed-length spins. As is well known, this model exhibits no spontaneous magnetisation, and we conclude that no interface of the Bloch-wall type exists at any finite temperature, but that the susceptibility discussed above is infinite for all $T < T_B$. In practice, however, any crystalline system has residual anisotropies in the transverse directions. The effect of anisotropies on the planar model has been investigated in some detail by José *et al* (1977), and we may draw on their results for some qualitative conclusions. Firstly, the case of a single preferred transverse direction should be essentially the same as the case $n = 1$ considered above. For square anisotropy, we expect a transition at temperature T_B between Bloch-like and Ising-like profiles, but with non-universal exponents, depending on the magnitude of the residual anisotropy. For hexagonal anisotropy, we expect the transition to take place in two stages. Below a lower critical temperature, T'_B, the profile should be Bloch-like, the transverse magnetisation pointing along one of the six preferred directions. For $T'_B < T < T_B$, there should be an infinite susceptibility but no transverse magnetisation. For $T > T_B$, the profile should be Ising-like, and the susceptibility should diverge as $T \to T_B^+$, again with a non-universal exponent.

3.3. Models with four or more components ($n \geq 3$).

We discuss this case briefly, for the sake of completeness, although we are not aware of any physical system to which it might be relevant. Although physical realisations of models with four or more components exist (Mukamel and Krinsky 1976) they exhibit antiferromagnetic or structural phase transitions, and would not be expected to have domain walls. With perfect transverse isotropy, there would be an Ising-like profile at all temperatures with no singularities, except possibly at $T = 0$. In the presence of residual transverse anisotropies, various types of transition, including those discussed above, would be possible, depending on the nature of the anisotropy.

Finally, it should be remarked that the presence of dipolar interactions, which are not included in our model, is likely to be important in magnetic systems. We have not considered this question in detail but, as observed in the Introduction, the effect of a significant magnetostatic energy is probably to give an effective Hamiltonian of the two-component ($n = 1$) type.

Acknowledgments

We are indebted to Dr P Rhodes and Professor D J Wallace for illuminating discussions. The work of Sarker *et al* was drawn to our attention by Professor Wallace, who learned of it from Professor S E Trullinger.

Appendix

The analogues in the Bloch-like region of the soft nodes (2.8) and (2.9) are eigenvectors of the operator

$$\begin{pmatrix} D_1 + 2(r-4g)t^2 & 2r^{1/2}(r-4g)^{1/2}st \\ 2r^{1/2}(r-4g)^{1/2}st & D_2 - \sigma + 2(r-4g)s^2 \end{pmatrix} \tag{A1}$$

which arises from the substitution (2.1) with (1.8) and (1.9). The operators D_1 and D_2 are given by (2.4) and (2.5), while s and t denote sech λz and tanh λz with $\lambda^2 = 2g$. One easily verifies that the translation mode $(\nabla M_1, \nabla M_2)\exp(\mathrm{i}\underset{\sim}{k}\cdot\underset{\sim}{x})$ is an eigenvector with eigenvalue $\mu_0 = k^2$. We now show that the second soft mode and its eigenvalue have the form

$$u_0 = \sum_n s^n \begin{matrix} a_n\lambda z + b_n t \\ c_n + d_n\lambda z t \end{matrix}; \qquad \mu_\sigma = 2(r-4g) + k^2. \tag{A2}$$

The distribution of even and odd functions guarantees orthogonality to the translation mode, and it clearly suffices to consider the case $k = 0$.

Substitution of (A2) in the eigenvalue equation yields the following recursion relations for the coefficients a_n, b_n, c_n and d_n:

$$|2(r-4g) - \mu - \lambda^2(n^2-4)|a_n$$
$$= |2(r-4g) - \lambda^2(n^2-3n-4)|a_{n-2} + 2r^{1/2}(r-4g)^{1/2}(d_{n-3} - d_{n-1}), \tag{A3}$$

$$|2(r-4g) - \mu - \lambda^2(n^2-4)|b_n + 2n\lambda^2 a_n$$
$$= |2(r-4g) - \lambda^2(n^2-n-6)|b_{n-2} - 2r^{1/2}(r-4g)^{1/2}c_{n-1}, \tag{A4}$$

$$|\lambda^2(n^2-1) + \mu|c_n - 2n\lambda^2 d_n$$
$$= |2(r-4g) + \lambda^2 n(n-3)|c_{n-2}$$
$$+ 2r^{1/2}(r-4g)^{1/2}(b_{n-1} - b_{n-3}) - 2\lambda^2(n-1)d_{n-2}, \tag{A5}$$

$$|\lambda^2(n^2-1) + \mu|d_n$$
$$= |2(r-4g) + \lambda^2(n^2-n-2)|d_{n-2} + 2r^{1/2}(r-4g)^{1/2}a_{n-1}. \tag{A6}$$

Since (A2) is to be a normalised bound state, these relations must be solved with the initial conditions $a_n = b_n = c_n = d_n = 0$ for $n \leqslant 0$. Evidently (A3) and (A6) form a closed system which may be solved independently. Setting $n = 1$ in (A6) yields

$$\mu d_1 = 0 \tag{A7}$$

with the given initial conditions. If this equation is satisfied by taking $\mu = 0$, the complete solution can be found in closed form. For $z > 0$, it is identical with the translation mode, but the parity of (A2) must be maintained by suitable sign changes at the origin, and we conclude that this solution is not admissible as an independent eigenvector. We therefore take $d_1 = 0$, and proceed to set $n = 2$ in (A3). We then obtain a non-trivial solution provided that the eigenvalue μ is given exactly by

$$\mu = 2(r-4g), \tag{A8}$$

which yields the desired result (A2). One may then find successive coefficients a_n and d_n in terms of the undetermined parameter a_2. In order to find a suitable value for a_2, consider (A4) with $n = 2$. This gives

$$4\lambda^2 a_2 = -2r^{1/2}(r-4g)^{1/2}c_1. \tag{A9}$$

The value

$$c_1 = r^{1/2} \tag{A10}$$

must be fixed by requiring that u_σ reduce to (2.9) when $r=4g$. Although the determination of successive coefficients does not lead to a power series in $(r-4g)^{1/2}$, one easily verifies that the remaining terms are at least of order $(r-4g)$.

References

Aharony A 1976 in *Phase Transitions and Critical Phenomena* vol 6, ed. C Domb and M S Green (London: Academic)
Amit D J 1978 *Field Theory, the Renormalization Group and Critical Phenomena* (London: McGraw-Hill)
Amit D J and Goldschmidt Y 1978 *Ann. Phys., NY* **114** 356–409
Diehl H W, Kross D M and Wagner H 1980 *Z. Phys.* B **36** 329–33
Fisher M E and Pfeuty P 1972 *Phys. Rev.* B **6** 1889–91
Gervais J L, Jevicki A and Sakita B 1976 *Phys. Rev.* D **12** 1038–51
Gervais J L and Sakita B 1975 *Phys. Rev.* D **11** 2943–9
Huang J S and Webb W W 1969 *J. Chem. Phys.* **50** 3677–93
José J, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217–41
Lajzerowicz J and Niez J J 1979 *J. Physique Lett.* **40** 165–9
Lowe M J and Wallace D J 1980 *Phys. Lett.* **93B** 433–6
Mukamel D and Krinsky S 1976 *Phys. Rev.* B **13** 5065–84
Nelson D R and Domany E 1976 *Phys. Rev.* B **13** 236–50
Ohta T and Kawasaki K 1977 *Prog. Theor. Phys.* **58** 467–81
Rudnick J and Jasnow D 1978a *Phys. Rev.* B **17** 1351–4
—— 1978b *Phys. Rev. Lett.* **41** 698–701
Sarker S, Trullinger S E and Bishop A R 1976 *Phys. Lett.* **59A** 255–8
Wallace D J and Zia R K P 1979 *Phys. Rev. Lett.* **43** 808–12
Wilson K G 1972 *Phys. Rev. Lett.* **28** 548–51
Wu E S and Webb W W 1973 *Phys. Rev.* A **8** 2065–76